Technical Note

Full-Range Solution for the Theis Well Function

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Abstract: Pumping tests are used to determine the transmissivities and storage coefficients of aquifers. For nonleaky aquifers, the solution reduces to the exponential integral, which is also called the Theis well function. It is beneficial to have a simple and handy approximation for the Theis well function. A simple (minimum number of terms) and reliable approximation that is efficient, yet sufficiently accurate, is preferable. This research provides a simple and accurate approximation to the Theis well function valid for all values of its arguments. The approximation is constructed by a combination of two solutions that are valid for small and large arguments. The approximation contains unknown coefficients, which are determined by using an optimization procedure. An accurate approximation to the well function should also be able to accurately compute the derivative of the well function. The proposed approximation minimizes errors in both the well function and its derivative. The maximum relative error in the proposed approximation and its derivative are less than 0.2% and 0.22%, respectively, making it useful for routine groundwater applications. **DOI: 10.1061/(ASCE)HE.1943-5584.0000833.** *© 2014 American Society of Civil Engineers.*

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Introduction

The exponential integral has many applications in transient groundwater flow, hydrological problems, mathematical physics, and applied mathematics. It is widely employed in the Theis (1935) solution, a key equation used for pumping test analysis in groundwater engineering. The Theis solution is used to determine the transmissivity and storativity of nonleaky aquifers. Computation of the exponential integral or Theis well function, with the conventional notation $W(u)$, is an essential task for this process. The function may be used alone or superposed in models with multiple wells, variable discharge, or image wells (Fitts 2012). The exponential integral, also called the well function in this context, is defined in the range of $u > 0$ by

$$W(u) = \int_u^\infty \frac{e^{-x}}{x} dx \quad u > 0 \tag{1}$$

There is a singularity in the integrand at zero. The upper incomplete gamma function, $\Gamma(a, u)$, reduces to the well function for $a = 0$, that is, $W(u) = \Gamma(0, u)$. Computing the gamma function can be implemented using a commercial mathematical software (such as *Mathematica*, *MathCad*, and *Maple*) or free internet sites (Wolfram Alpha 2013). One can get accurate values of the well function in Wolfram Alpha by using the command Gamma[0, u]. Commercial mathematical software packages are not always available. Moreover, computational speed (CPU time) for nonlinear systems should be accounted for when the intensive computation of the well function (e.g., in optimization problems) is required. In this case, free internet sites are inappropriate because of the time required for calculations.

As shown in Fig. 1, the well function behaves like a negative exponential for large values of the argument and like a logarithm for small values of the argument. This behavior of Eq. (1) makes developing a full-range straightforward approximation to calculate the well function a difficult task.

Numerical approximations of the well function have been widely investigated in the literature. Tseng and Lee (1998) reported a complete survey of these approximations. Most of the previous studies used polynomial and rational approximations, infinite and truncated series expansions, or asymptotic expansions that are valid in a limited range of the argument u (e.g., Allen 1954; Spiegel 1968; Cody and Thacher 1968; Abramowitz and Stegun 1970; Hamming 1973; Beyer 1978; Srivastava 1995; Tseng and Lee 1998; Srivastava and Guzman-Guzman 1998). The aforementioned researchers proposed approximate expressions to compute the well function over specific ranges of the argument. An asymptotic (divergent) series may be valid for large values of u. Among all these approximations, the series expansion based on expanding the exponential function in Eq. (1) in power series and integrating term by term is still the most widely used one for small values of u (Cody and Thacher 1968; Stegun and Zucker 1974). For small-u, series representation is $W(u) = -0.5772 - \ln(u) + u - u^2/(2 \times 2!) + u^3/(3 \times 3!) - u^4/(4 \times 4!) + \cdots$.

In practical applications, we often need to approximate $W(u)$ for a wide range of u. Due to the lack of an accurate approximation over the full range of u, an entire set of pump test data is rarely used in analysis (Swamee and Ojha 1990). An approximation that is efficient and satisfies the required accuracy is still highly preferable and any effort for presenting a simple full-range solution with reasonable accuracy would be of practical importance. The main focus of the current study is on a simple and accurate approximation to $W(u)$ that is valid over the entire practical range of u. This straightforward approximation should be expressed in terms of common mathematical functions such as logarithmic and exponential functions and preferably should have integer powers allowing for faster computations.

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J. Hydrol. Eng. 2014.19:649-653.

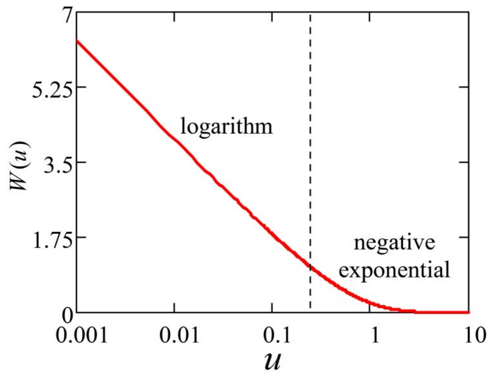

Fig. 1. Well function curve

Existing Full-Range Approximations

There are few approximations that are valid for the entire range of the argument. Swamee and Ojha (1990) first obtained an approximation for $W(u)$ valid for all positive u as

$$W_*(u)=\left(\left\{\ln\left[\left(\frac{0.56146}{u}+0.65\right)(1+u)\right]\right\}^{-7.7}+u^4e^{7.7u}(2+u)^{3.7}\right)^{-0.13} \quad (2)$$

where $W_*(u)$ is the approximation to $W(u)$ of Eq. (1).

Srivastava (1995) showed that Eq. (2) resulted in overprediction of the storage coefficient, especially for an aquifer with a low storage coefficient or high transmissivity. He also stated that the derivative of the drawdown curve became more critical than the drawdown for slope-matching methods, and thus an accurate approximation of the well function should be able to accurately reproduce the derivative of the well function.

The percentage error, PE(%), of the well function approximation can be determined as

$$\mathrm{PE}(\%)=100\times\frac{W(u)-W_*(u)}{W(u)} \quad (3)$$

where $W_*(u)$ stands for the proposed algebraic approximation of the well function and $W(u)$ stands for the accurate numerical results of the well function. The accurate numerical results can be determined using Eq. (1) for the known value of u with the aid of popular software such as *Maple*, *Mathematica*, *Matlab*, or *Mathcad*. The final results with ten significant digits are used in this study.

As shown in Fig. 2, for the practical range of $0 < u < 10$, the maximum percentage error [PE(%)], of Eq. (2) and its derivative [analytical differentiation of Eq. (1)], compared to the exact value of the derivative $-e^{-u}/u$, are 1.28 and 2.1%, respectively. As will be shown, the error in Eq. (2) is even more pronounced when using this approximation in a realistic application.

Barry et al. (2000) also offered a full-range approximation of the well function as

$$W_*(u)=\frac{e^{-u}\ln\left[1+\frac{0.5615}{u}-0.4385\left(1.0421u+\frac{1}{1+u^{1.5}}+\frac{1.0801}{1+2.35u^{-1.0919}}\right)^{-2}\right]}{0.5616+0.4385e^{-2.2803u}} \quad (4)$$

This analytical approximation is constructed by interpolation between the exponential integral's small and large asymptotes. As shown in Fig. 3, for the practical range of $0 < u < 100$, the maximum percentage error of Eq. (4) and its derivative, are 0.07 and 0.2%, respectively. Eq. (4) is accurate enough but is not very simple. As will be shown in the next section, a simpler approximation to the well function can be derived with the same accuracy as Eq. (4).

Proposed Full-Range Approximation

For small values of u, the well function can be approximated by

$$W_*(u)=(1+a_1u^{a_2})\ln\left(\frac{a_3}{u}+a_4\right) \quad (5)$$

Also, for large values of u, the well function can be approximated as

$$W_*(u)=\frac{1}{ue^u}\left(\frac{u+a_5}{u+a_6}\right) \quad (6)$$

Eq. (5) is a modified form of logarithmic term in Eq. (2) presented by Swamee and Ojha (1990). Eq. (6) is also a modified truncated version of the approximation presented by Allen (1954) and Abramowitz and Stegun (1970), which was firstly proposed by Srivastava and Guzman-Guzman (1998) by considering $a_5 = 0.3637$ and $a_6 = 1.282$.

Eqs. (5) and (6) can be combined to yield the following approximation of the well function agreeing with all values of the argument:

$$W_*(u)=\left\{\left[(1+a_1u^{a_2})\ln\left(\frac{a_3}{u}+a_4\right)\right]^{-P}+\left[\frac{1}{ue^u}\left(\frac{u+a_5}{u+a_6}\right)\right]^{-P}\right\}^{-1/P} \quad (7)$$

in which constant coefficients P and a_i ($i = 1, 2, \ldots, 6$) will be determined via an optimization procedure. As mentioned before, an accurate approximation of the well function should be able to accurately reproduce the derivative of the well function. Differentiating both sides of Eq. (7) with respect to u yields

J. Hydrol. Eng. 2014.19:649-653.

$$\frac{dW_*(u)}{du} = \frac{\left\{\frac{a_1 a_2 u^{a_2-1}\ln\left(\frac{a_3}{u}+a_4\right) - \frac{a_3(1+a_1u^{a_2})}{u^2\left(\frac{a_3}{u}+a_4\right)}}{\left[(1+a_1u^{a_2})\ln\left(\frac{a_3}{u}+a_4\right)\right]^{P+1}}\right\} - \left[\frac{\frac{u+a_5}{u+a_6} - \frac{a_6-a_5}{(u+a_6)^2} + \frac{1}{u}\left(\frac{u+a_5}{u+a_6}\right)}{\frac{1}{u^P e^{Pu}}\left(\frac{u+a_5}{u+a_6}\right)^{P+1}}\right]}{\left\{\left[(1+a_1u^{a_2})\ln\left(\frac{a_3}{u}+a_4\right)\right]^{-P} + \left[\frac{1}{ue^u}\left(\frac{u+a_5}{u+a_6}\right)\right]^{-P}\right\}^{1/P+1}} \quad (8)$$

To determine P and a_i, the sum of the maximum absolute percentage error in both well function approximation, Eq. (7), and its derivative, Eq. (8), was minimized as an objective function using the Solver toolbox of Microsoft Excel. For this, different values of u ranging from 0.001 to 100 were used and the coefficients that minimized the sum of the maximum absolute percentage errors, $(\mathrm{Max}|[W(u) - W_*(u)]/W(u)| + \mathrm{Max}|[dW(u)/du - dW_*(u)/du]/dW(u)/du|)$ were determined. The final results are $P = 2$, $a_1 = -0.19$, $a_2 = 0.7$, $a_3 = 0.565$, $a_4 = 4$, $a_5 = 0.444$, and $a_6 = 1.384$. Substituting these values into Eq. (7) yields

$$W_*(u) = \left\{\frac{(1-0.19u^{0.7})^{-2}}{\left[\ln\left(\frac{0.565}{u}+4\right)\right]^2} + u^2 e^{2u}\left(\frac{u+1.384}{u+0.444}\right)^2\right\}^{-1/2} \quad (9)$$

Eq. (9) can be used for the entire practical range of the argument. Over the range $0.001 < u < 100$, the maximum relative error associated with Eq. (9), compared to the precise numerical results, and its derivative, compared to the exact value of the derivative $-e^{-u}/u$, are 0.2 and 0.22%, respectively, as shown in Fig. 4.

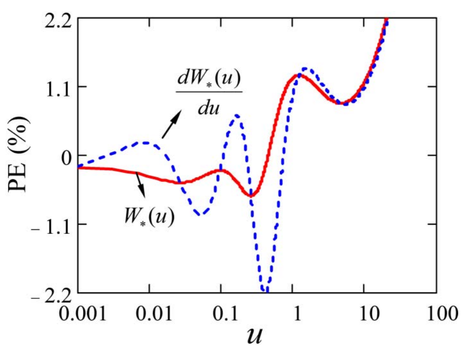

Fig. 2. Percentage error of approximation proposed by Swamee and Ojha (1990)

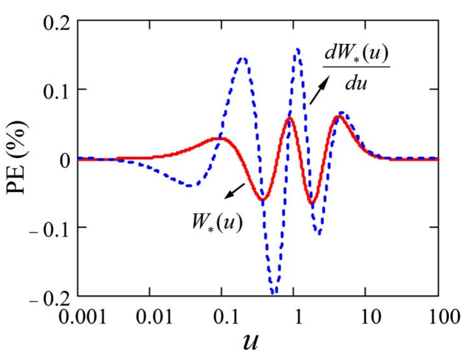

Fig. 3. Percentage error of approximation proposed by Barry et al. (2000)

Table 1 presents a summary of the existing full-range solutions and the one proposed in this research. This table clearly indicates that the solution proposed in this study offers both simplicity and accuracy.

Practical Application (Discrete Kernel Generator)

A groundwater management application involving the well function is useful as it shows the likely error involved in using the well function approximation in a realistic application. The discrete pumping kernel, U, defined as (Morel-Seytoux and Daly 1975)

$$U = \frac{1}{4\pi T}\left\{W\left(\frac{r^2 S}{4tT}\right) - W\left[\frac{r^2 S}{4(t-\tau)T}\right]\right\} \quad (10)$$

where T is the transmissivity, S is the storativity, t is the time ($t = 0$ when well turns on, and $t = \tau$ when well turns off), τ is a unit time period, and r is the distance from the well. Eq. (10) applies for $t > \tau$ (after well turns off). This equation involves subtraction, thus possible errors of this equation may increase compared with a single approximation of the well function. To show the accuracy obtainable with the full-range approximations, consider the cases examined by Tseng and Lee (1998) and Barry et al. (2000). The percentage errors of Eq. (10), using the proposed full-range approximations of the well function [Eqs. (2), (4), and (9)] are computed for $T = 10{,}000\ \mathrm{m^2/week}$, $S = 0.2$, $\tau = 1$ week, and different values of the distance from the well, r, as shown in Fig. 5. For this test case, u is in the range of $0.31 < u < 72$. As can be observed in the figure, the proposed approximations by Eqs. (4) and (9) work well over the whole range of the argument in this realistic example (the percentage error of the computed results is less than 0.22%). However for Eq. (2), the errors become meaningful (about 3.8%). This example shows that Eq. (2) is not very accurate for practical purposes as argued by Srivastava (1995). Both Eqs. (4) and (9) are accurate enough for most applications. Eq. (4) has slightly more consistent accuracy, but Eq. (9) is simpler and faster computationally.

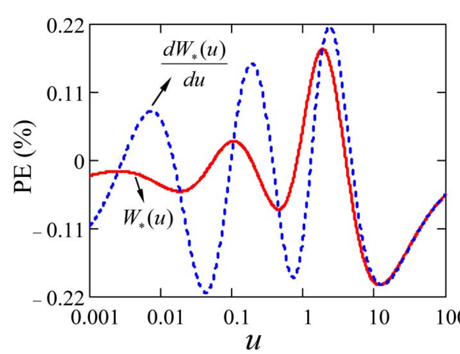

Fig. 4. Percentage error of proposed approximation in current research

J. Hydrol. Eng. 2014.19:649-653.

Table 1. Summary of Full-Range Approximations for Well Function in Confined Aquifers

Investigator	Proposed approximation $W_*(u)$	Maximum error (%)	Maximum error of derivative (%)
Swamee and Ojha (1990)	$\{(\ln[(0.56146/u+0.65)(1+u)])^{-7.7}+u^4e^{7.7u}(2+u)^{3.7}\}^{-0.13}$	1.28	2.1
Barry et al. (2000)	$[e^{-u}/(0.5615+0.4385e^{-2.2803u})]\times\ln\{1+0.5615/u$ $-0.4385[1.0421u+1/(1+u^{1.5})+1.0801/(1+2.35u^{-1.0919})]^{-2}\}$	0.07	0.2
Current research	$\{(1-0.19u^{0.7})^{-2}/[\ln(0.565/u+4)]^2+u^2e^{2u}(u+1.384)^2/(u+0.444)^2\}^{-0.5}$	0.2	0.22

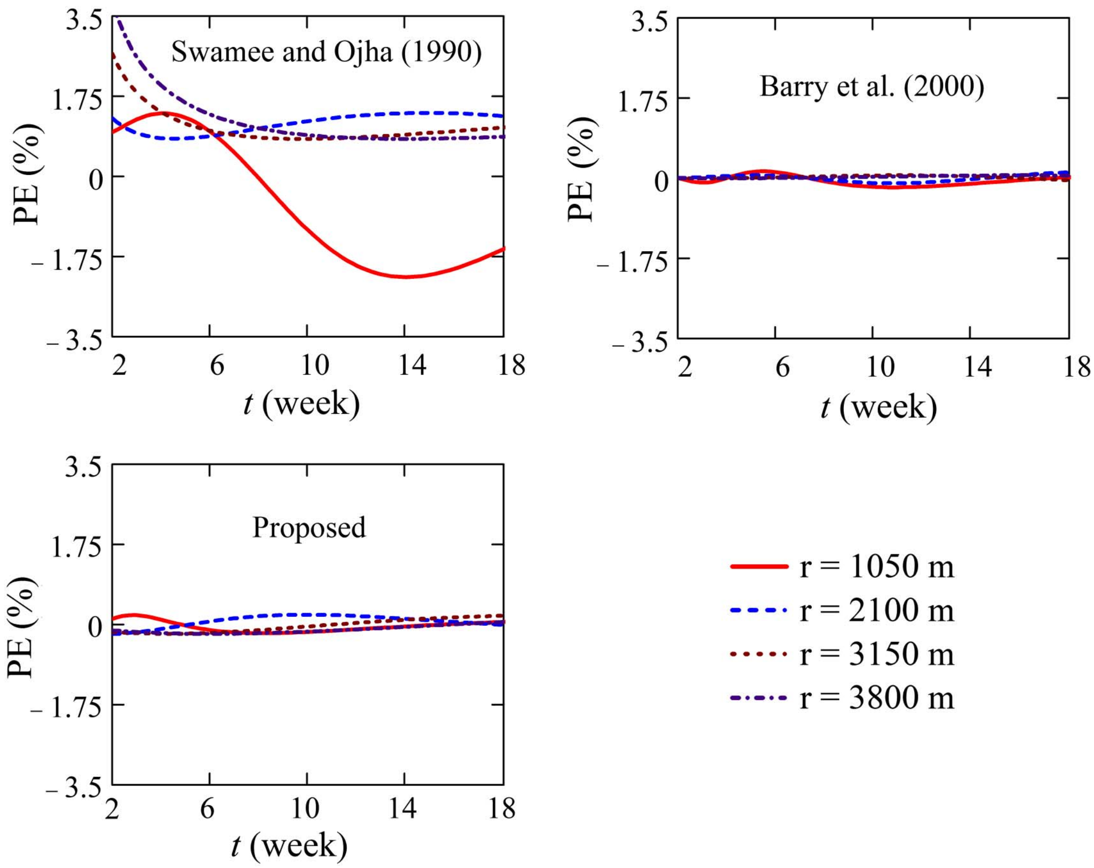

Fig. 5. Percentage error of unit pulse responses, U (week/m^2), at selected distances from the pumping well, $r(m)$, as a function of the time t (week)

Conclusion

An empirical well function valid for the full range of argument is developed for confined aquifers. The maximum relative error of the proposed equation is less than 0.2%. This approximation of the well function is also able to reproduce the derivative of the well function with a maximum relative error less than 0.22%. The proposed approximation exhibits both simplicity and accuracy. Due to the simple form and acceptable accuracy of the proposed approximation [Eq. (9)], it is proposed in lieu of previously proposed equations for approximating the well function in confined aquifers. The approximation developed in this research can be easily implemented in software algorithms, calculators, or spreadsheets. The author hopes that the efficient equation presented in this study is useful in groundwater and hydrogeological applications.

Notation

The following symbols are used in this paper:

e = Neper's number;
r = distance from the well;
S = storativity;
T = transmissivity;
t = time;
u = argument of the well function;
$W(u)$ = well function;
$W_*(u)$ = approximation of $W(u)$;
$\Gamma(a,u)$ = upper incomplete gamma function; and
τ = unit time period.

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J. Hydrol. Eng. 2014.19:649-653.